Superposition of macroscopically distinct states means large multipartite entanglement

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We show relations between superposition of macroscopically distinct states and entanglement. These relations lead to the important conclusion that if a state contains superposition of macroscopically distinct states, the state also contains large multipartite entanglement in terms of several measures. Such multipartite entanglement property also suggests that if a state contains superposition of macroscopically distinct states, a measurement on a single particle drastically changes the state of macroscopically many other particles, as in the case of the N-qubit GHZ state.

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I. INTRODUCTION

Superposition of macroscopically distinct states is one of the most fundamental characteristics in quantum physics, which has been attracting much attentions since the birth of quantum physics [1]. On the experimental side, such superposition has been explored in various many-body systems, such as magnetic materials [2] and trapped ions [3]. On the theoretical side, useful quantities and methods have been proposed in order to quantitatively investigate such superposition [4, 5, 6, 7, 8, 9, 10]. For example, Leggett introduced the criterion, "disconnectivity", and studied superposition of macroscopically distinct states in superconducting systems [4]. Mermin proposed a many-partite Bell-like inequality, and showed that it is exponentially violated for the N-qubit GHZ state $|\mathrm{GHZ}\rangle \equiv (|0^{\otimes N}\rangle + |1^{\otimes N}\rangle)/\sqrt{2}$, which is a typical example of superposition of macroscopically distinct states [5].

Entanglement is the other important property of quantum states. After being introduced by Schrödinger [1], a great deal of research has been performed on the physical and mathematical characteristics of entanglement itself [11] and on the applications of entanglement to condensed matter physics [12]. Nowadays, entanglement is also known as a crucial resource for quantum information processing [13].

Since both superposition of macroscopically distinct states and entanglement represent "quantumness" of physical systems, it is reasonable to expect that there are some fundamental relations between them. However, to the author's knowledge, such relations have so far eluded us. The purpose of this Rapid Communication is to show several relations between superposition of macroscopically distinct states and entanglement. From these relations, we obtain the important conclusion that a state which contains superposition of macroscopically distinct states also contains large multipartite entanglement in terms of several measures, such as the localizable entanglement [14], the distance-like measure of entanglement [15], and multipartite entanglement defined through various bipartitions.

Furthermore, such large multipartite entanglement property also leads to another interesting consequence. As is well known, the projective measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on a single particle of the N-qubit GHZ state drastically changes the state of other $N-1$ particles. Since the N-qubit GHZ state is a typical example of superposition of macroscopically distinct states, it is legitimate to think that other states which contain superposition of macroscopically distinct states also have such property. In fact, we will show that it is the case.

II. INDEX p

As the criterion of superposition of macroscopically distinct states, we use index p [9]. Let us consider an N-site lattice ($1 \ll N < \infty$) where the dimension of the Hilbert space on each site is an N-independent constant, such as a chain of N spin-1/2 particles. Since we are interested in the macroscopic properties of the system, we use two symbols, O and o, in order to describe asymptotic behaviors of a function $f(N)$ in the thermodynamic limit $N \to \infty$: $f(N) = O(N^k)$ means $\lim_{N\to\infty}[f(N)/N^k] = \text{const.} \neq 0$ and $f(N) = o(N^k)$ means $\lim_{N\to\infty}[f(N)/N^k] = 0$.

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For a given pure state $|\psi\rangle$, the index p ($1 \le p \le 2$) is defined by

$$\max_{\hat{A}} C(\hat{A}, \hat{A}, |\psi\rangle) = O(N^p),$$

where $C(\hat{X}, \hat{Y}, |\psi\rangle) \equiv \langle\psi|\hat{X}\hat{Y}|\psi\rangle - \langle\psi|\hat{X}|\psi\rangle\langle\psi|\hat{Y}|\psi\rangle$ is the correlation, and the maximum is taken over all Hermitian additive operators $\hat{A}$. Here, an additive operator $\hat{A} = \sum_{l=1}^{N} \hat{a}(l)$ is a sum of local operators $\{\hat{a}(l)\}_{l=1}^{N}$, where $\hat{a}(l)$ is a local operator acting on site l. For example, if the system is a chain of N spin-1/2 particles, $\hat{a}(l)$ is a linear combination of three Pauli operators, $\hat{\sigma}_x(l), \hat{\sigma}_y(l), \hat{\sigma}_z(l)$, and the identity operator $\hat{1}(l)$ acting on site l. In this case, the x-component of the total magnetization $\hat{M}_x \equiv \sum_{l=1}^{N} \hat{\sigma}_x(l)$ and the z-component of the total staggard magnetization $\hat{M}_z^{st} \equiv \sum_{l=1}^{N} (-1)^l \hat{\sigma}_z(l)$ are, for example, additive operators. The index p takes the minimum value 1 for any product state $\bigotimes_{l=1}^{N} |\phi_l\rangle$, where $|\phi_l\rangle$ is a state of site l (this means that $p > 1$ is an entanglement witness for pure states). If p takes the maximum value 2, the state contains superposition of macroscopically distinct states, because in this case a Hermitian additive operator has a "macroscopically large" fluctuation in the sense that the relative fluctuation does not vanish in the thermodynamic limit: $\lim_{N\to\infty} \left[C(\hat{A}, \hat{A}, |\psi\rangle)^{1/2}/N \right] \neq 0$, and because the fluctuation of an observable in a pure state means the existence of a superposition of eigenvectors of that observable corresponding to different eigenvalues. For example, the N-qubit GHZ state $|\mathrm{GHZ}\rangle \equiv (|0^{\otimes N}\rangle + |1^{\otimes N}\rangle)/\sqrt{2}$, which obviously contains superposition of macroscopically distinct states, has $p = 2$, since $C(\hat{M}_z, \hat{M}_z, |\mathrm{GHZ}\rangle) = N^2$.

III. BIPARTITE ENTANGLEMENT

Let us first briefly examine relations of p to bipartite entanglement. One of the simplest ways of quantifying bipartite entanglement in a quantum many-body system is to divide the total system into two equal-size subsystems, e.g., dividing $\{1, 2, ..., N\}$ into $\{1, 2, ..., N/2\}$ and $\{N/2+1, ..., N\}$, and calculate the von Neumann entropy E ($0 \le E \le N/2$) of the reduced density operator of one subsystem. For translationally invariant systems, which are ubiquitous in condensed matter physics, this is a good way of evaluating bipartite entanglement. It is easy to show that the BEC of magnons (or $N/2$-Dicke state): $\left(\sum_{l=1}^{N} |1\rangle_l \langle 0| \right)^{N/2} |0^{\otimes N}\rangle$ has $p = 2$ whereas the entanglement entropy E is as small as $O(\log N)$ ($\ll N/2$). Furthermore, it is immediate to show $E = \log 2$ for the N-qubit GHZ state which has $p = 2$. Therefore, superposition of macroscopically distinct states does not necessarily mean large bipartite entanglement in terms of this measure.

The other way of quantifying bipartite entanglement in a quantum many-body system is to choose two representative sites and evaluate the concurrence [16] between them. This approach is often taken in the study of entanglement behavior at quantum critical points. It is easy to calculate that the concurrence between any two sites of the N-qubit GHZ state is 0, whereas that of the BEC of magnons is $O(1/N)$ (note that $2/N$ is the maximum value of the concurrence for homogeneous systems [17]). Since these two states both have $p = 2$, there is no direct connection between p and concurrence (and hence the entanglement of formation).

In short, we conclude that there is no significant relation between superposition of macroscopically distinct states and bipartite entanglement. In particular, superposition of macroscopically distinct states does not necessarily mean large bipartite entanglement. In the followings, we next consider multipartite entanglement.

IV. LOCALIZABLE ENTANGLEMENT

The localizable entanglement between two sites is defined by the maximum amount of entanglement that can be localized in these two sites, on average, by doing local measurements on other sites [14].

We can show that if $p = 2$, the state has large multipartite entanglement in the sense that macroscopically many ($O(N^2)$) pairs of sites have non-vanishing amount ($O(1)$) of localizable entanglement. In order to show it, let $S \equiv \{1, 2, ..., N\}$ be the set of all sites, $|\psi\rangle$ be the state of the total system, and $\hat{A} = \sum_{l=1}^{N} \hat{a}(l)$ be an additive operator. Let us define two subsets

$$\begin{aligned} R_1 &\equiv \{(l, l') \in S \times S \mid C(\hat{a}(l), \hat{a}(l'), |\psi\rangle) = O(1)\} \\ R_2 &\equiv \{(l, l') \in S \times S \mid C(\hat{a}(l), \hat{a}(l'), |\psi\rangle) = o(1)\} \end{aligned}$$

of $S \times S$. In other words, R_1 is the set of pairs such that the correlation persists in the thermodynamic limit, and R_2

is that of others. Let us assume that the number $|R_1|$ of elements of R_1 is $o(N^2)$ for any $\hat{A}$. Then,

$$\begin{aligned} C(\hat{A},\hat{A},|\psi\rangle) &= \sum_{l=1}^{N}\sum_{l'=1}^{N} C(\hat{a}(l),\hat{a}(l'),|\psi\rangle) \\ &= \Big[\sum_{(l,l')\in R_1} + \sum_{(l,l')\in R_2}\Big] C(\hat{a}(l),\hat{a}(l'),|\psi\rangle) \\ &\le o(N^2)O(1) + O(N^2)o(1) = o(N^2) \end{aligned}$$

for any $\hat{A}$, which means $p<2$. Therefore, if $p=2$, $|R_1| = O(N^2)$ for an $\hat{A}$.

It is known that the localizable entanglement between two sites is lower bounded by the maximum correlation between these two sites [14]. Therefore, the above result means that if $p=2$, macroscopically many ($O(N^2)$) pairs have finite ($O(1)$) amount of localizable entanglement in the thermodynamic limit. In this sense, a state which contains superposition of macroscopically distinct states has large multipartite entanglement.

V. DISTANCE-LIKE MEASURE OF ENTANGLEMENT

In Ref. [15], a measure of multipartite entanglement for $\hat{\rho}$ was defined by

$$E_D(\hat{\rho}) \equiv \min_{\hat{\sigma}} D(\hat{\rho},\hat{\sigma}).$$

Here, D is a distance in the Hilbert space and the minimum is taken over all separable state $\hat{\sigma} \equiv \sum_i \lambda_i \bigotimes_{l=1}^{N} |\phi_l^{(i)}\rangle\langle\phi_l^{(i)}|$, where $0 \le \lambda_i \le 1$, $\sum_i \lambda_i = 1$, and $|\phi_l^{(i)}\rangle$ is a state of site l.

In order to clarify the relation between superposition of macroscopically distinct states and E_D, it is necessary to consider mixed states. However, $p=2$ is not a witness of superposition of macroscopically distinct states in mixed states, since a fluctuation is not necessarily equivalent to a coherence in mixed states (for example, consider the state $|0^{\otimes N}\rangle\langle 0^{\otimes N}| + |1^{\otimes N}\rangle\langle 1^{\otimes N}|$, which has macroscopically large fluctuation but no coherence).

Index q, which was introduced in Ref. [10], is a criterion of superposition of macroscopically distinct states in mixed states. For a given many-body state $\hat{\rho}$, index q ($1 \le q \le 2$) is defined by

$$\max\Big(N, \max_{\hat{A}} \Big\|[\hat{A},[\hat{A},\hat{\rho}]]\Big\|_1\Big) = O(N^q),$$

where $\|\hat{X}\|_1 \equiv \mathrm{Tr}\sqrt{\hat{X}^\dagger \hat{X}}$ is the 1-norm, and $\max_{\hat{A}}$ means the maximum over all Hermitian additive operators $\hat{A}$. As detailed in Ref. [10], q takes the minimum value 1 for any separable state, and if q takes the maximum value 2, the state contains superposition of macroscopically distinct states. In particular, for pure states, $p=2 \iff q=2$.

Let $\hat{\sigma}$ be a separable state and $\hat{\rho}$ be a state having $q=2$. Then,

$$\begin{aligned} O(1) &= \frac{\big\|[[\hat{\rho},\hat{A}],\hat{A}]\big\|_1 - \big\|[[\hat{\sigma},\hat{A}],\hat{A}]\big\|_1}{N^2} \\ &\le \frac{\big\|[[\hat{\rho}-\hat{\sigma},\hat{A}],\hat{A}]\big\|_1}{N^2} \le 4\|\hat{\rho}-\hat{\sigma}\|_1 \end{aligned}$$

for an additive operator $\hat{A}$. Therefore, if we choose the 1-norm as the distance D, we obtain $E_D(\hat{\rho}) = O(1)$, which means that if $q=2$ the state has persistent multipartite entanglement in the thermodynamic limit.

The Bures distance and the relative entropy are often used as the distance in E_D. The Bures distance is defined by $\sqrt{2(1-\|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|_1)}$. Then, we can show that if $q=2$, $E_D(\hat{\rho})$ is as large as $O(1)$, since $1-\|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|_1 = (1-\|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|_1^2)/(1+\|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|_1) \ge \frac{1}{8}\|\hat{\rho}-\hat{\sigma}\|_1^2 = O(1)$ [13]. On the other hand, the relative entropy is defined by $S(\hat{\rho}\|\hat{\sigma}) \equiv \mathrm{Tr}(\hat{\rho}\log\hat{\rho} - \hat{\rho}\log\hat{\sigma})$. By using the well-known inequality $S(\hat{\rho}\|\hat{\sigma}) \ge \frac{1}{2}\|\hat{\rho}-\hat{\sigma}\|_1^2$ [18], we again obtain $E_D(\hat{\rho}) \ge O(1)$ if $q=2$.

In summary, a state which contains superposition of macroscopically distinct states also has large multipartite entanglement in terms of the distance-like measures of entanglement [19].

VI. MULTIPARTITE ENTANGLEMENT DEFINED THROUGH VARIOUS BIPARTITIONS

Another way of evaluating multipartite entanglement in quantum many-body systems is to consider various bipartitions and evaluate bipartite entanglement among them. For example, Meyer and Wallach introduced the multipartite entanglement measure $2(1-\frac{1}{N}\sum_{l=1}^{N}\mathrm{Tr}(\hat{\rho}_l^2))$ by considering all bipartition between a single site and others, where $\hat{\rho}_l$ is the reduced density operator of site l [21]. Here, we consider similar multipartite entanglement.

Let $S\equiv\{1,2,...,N\}$ be the set of all sites and $|\psi\rangle_S$ be the state of the total system. Consider the Schmidt decomposition between site l and other sites, i.e., $S-l$:

$$|\psi\rangle_S=\sqrt{\lambda_0}|\xi_0\rangle_l|\phi_0\rangle_{S-l}+\sqrt{\lambda_1}|\xi_1\rangle_l|\phi_1\rangle_{S-l}.$$

Then, let us factor out, if any, the common state $|\omega\rangle$ in $|\phi_0\rangle_{S-l}$ and $|\phi_1\rangle_{S-l}$ as

$$|\psi\rangle_S \;=\; \Big(\sqrt{\lambda_0}|\xi_0\rangle_l|\eta_0\rangle_{S_1(l)}+\sqrt{\lambda_1}|\xi_1\rangle_l|\eta_1\rangle_{S_1(l)}\Big)|\omega\rangle_{S_2(l)}$$

so that the number $|S_2(l)|$ of sites in the subsystem $S_2(l)$ is maximum. Here,

$$\begin{aligned}|\phi_0\rangle_{S-l} &= |\eta_0\rangle_{S_1(l)}\otimes|\omega\rangle_{S_2(l)}\\ |\phi_1\rangle_{S-l} &= |\eta_1\rangle_{S_1(l)}\otimes|\omega\rangle_{S_2(l)}\end{aligned}$$

and $S_1(l)+S_2(l)=S-l$. Since the states of $l+S_1(l)$ is pure, entanglement between l and $S_1(l)$ is quantified by the entanglement entropy $E(l)$ $(0\le E(l)\le 1)$ as $E(l)=-\lambda_0\log_2\lambda_0-\lambda_1\log_2\lambda_1$.

Let us consider the quantity E_B $(0\le E_B\le N)$ defined by

$$E_B\equiv|\{l\in S\mid E(l)=O(1)\text{ and }|S_1(l)|=O(N)\}|,$$

where $|X|$ is the number of elements in the set X. $E(l)=O(1)$ means that entanglement between l and $S_1(l)$ does not vanish in the thermodynamic limit. $|S_1(l)|=O(N)$ means that site l is entangling with macroscopically many other sites. Therefore, E_B is the number of sites each of which is entangling with macroscopically many other sites with non-vanishing amount of entanglement. For example, the cluster state [22] has maximum entanglement in terms of $E_B=N$, since it is easily confirmed that $E(l)=1$ and $|S_1(l)|=N-1$ for any l.

We can show that if $|\psi\rangle_S$ has $p=2$, $E_B=O(N)$ (a proof is given in Appendix). This means that a state which contains superposition of macroscopically distinct states also has large multipartite entanglement in the sense of $E_B=O(N)$. Note that this also means the Meyer-Wallach's measure is large if $p=2$.

VII. EFFECT OF A MEASUREMENT ON A SINGLE PARTICLE

The advantage of considering E_B is that the effect of a measurement on a single particle becomes very clear. Let us randomly choose a single site, say site l, from S. From the above result, site l satisfies $E(l)=O(1)$ and $|S_1(l)|=O(N)$ with the non-vanishing probability $E_B/N=O(1)$ if $p=2$. The projective measurement $\{|\xi_0\rangle\langle\xi_0|,|\xi_1\rangle\langle\xi_1|\}$ on site l changes the state of $S_1(l)$ into $|\eta_0\rangle$ or $|\eta_1\rangle$, and this change is a "drastic" one since (i) the information gain through this measurement is as large as $E(l)=O(1)$ and (ii) the state of macroscopically many $|S_1(l)|=O(N)$ sites are changed by this measurement. Therefore, we obtain the second main conclusion that if a state contains superposition of macroscopically distinct states, a measurement on a single site drastically changes the state of macroscopically many other sites.

VIII. CONCLUSION AND DISCUSSION

In this Rapid Communication, we have shown relations between superposition of macroscopically distinct states and entanglement, and concluded that if a state contains superposition of macroscopically distinct states, the state also contains large multipartite entanglement in terms of several measures. We have also seen that if a state contains superposition of macroscopically distinct states, a projective measurement on a single particle drastically changes the state of macroscopically many other particles.

Since there are infinitely many measures for multipartite entanglement, and each measure sees different features of quantum many-body states, it is unrealistic to expect that superposition of macroscopically distinct states means large multipartite entanglement in terms of *any* measures. Indeed, if we say a state has a large multipartite entanglement

if macroscopically many particles are genuinely entangled [23], a superposition of macroscopically distinct states does not necessarily mean large multipartite entanglement, since a weak entanglement among macroscopically many particles is not enough for a state to have a macroscopic superposition (for example, the W-state). In the similar reasoning, we can conclude that a superposition of macroscopically distinct states does not necessarily mean large multipartite entanglement if the multipartite entanglement is defined by the minimum of the bipartite entanglement over all bipartitions [24].

Furthermore, if we consider cluster states [22], the discrepancy between large multipartite entanglement and macroscopic superposition becomes clear. It is known that a multipartite Bell's inequality [23], the Schmidt measure [25], and the geometric measure of entanglement [26] take large values for cluster states, whereas $p = 1$ for cluster states since they have no long-range two-point correlations. One of the reasons for this discrepancy seems to be the well known fact that the "quantumness" of cluster states is hidden in many-point correlations. In order to gain insight into this, let us consider a simple example, the RVB state: $|1,2\rangle|3,4\rangle...|N-1,N\rangle + |2,3\rangle|4,5\rangle...|N,1\rangle$, where $|i,j\rangle$ is the singlet between sites i and j. The RVB state is obviously the superposition of two macroscopically distinct VB states. However, the maximum entanglement between nearest-neighbor sites prohibits the existence of long-range two-point correlations in this state (i.e., entanglement monogamy). Therefore, the RVB state has $p = 1$. In spite of it, the RVB state has large multipartite entanglement in terms of several measures (for example, the measure E_B considered in this paper is as large as $O(N)$). It is known that in order to see quantum correlations in this state, at least four-point correlations are required.

The detailed analysis of the discrepancy between large multipartite entanglement and macroscopic superposition is, however, beyond the scope of the present paper. It is an important subject of the future study.

Finally, let us briefly discuss the relation of our results to the entanglement witness [23]. Experimental detections of multipartite entanglement is one of the most important subjects in today's quantum many-body physics, and many detection methods, i.e., witnesses, have been proposed [23]. Our results between superposition of macroscopically distinct states and multipartite entanglement imply that an experimental detection of macroscopic superposition is also a witness of multipartite entanglement in terms of several measures. An advantage of the detection of multipartite entanglement through the detection of macroscopic superposition is that it detects not only inseparability but also large multipartite entanglement. Among various entanglement witnesses, in particular, indices p and q are closely related to the witness through "collective measurements", such as the spin-squeezing parameter and the magnetic susceptibility [23], since p and q are defined by using additive operators. Because of the uncertainty relation, the squeezing of one component of the total magnetization leads to the large fluctuation of the other component. This large fluctuation represents the macroscopic superposition and large multipartite entanglement. On the other hand, the magnetic susceptibility is proportional to the fluctuation of the magnetization. The multipartite entanglement properties of a many-body ground state often gives long-range two-point correlations and therefore a large fluctuation of a component of the magnetization. This persists at sufficiently low temperature, and is detected through the measurement of the magnetic susceptibility.

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APPENDIX

Let $|\psi\rangle$ be the state of the total system S, and let us decompose $|\psi\rangle$ into a tensor product of inseparable states: $|\psi\rangle = \bigotimes_{i=1}^{r} |\psi_i\rangle$, where $|\psi_i\rangle$'s are inseparable states. We denote the subsystem corresponding to $|\psi_i\rangle$ by Z_i, i.e., $Z_1 + Z_2 + ... + Z_r = S$. Let us also decompose an additive operator $\hat{A} = \sum_{l=1}^{N} \hat{a}(l)$ according to this partition as $\hat{A} = \sum_{i=1}^{r} \hat{A}_i$, where $\hat{A}_i \equiv \sum_{l\in Z_i} \hat{a}(l)$ is an operator acting on Z_i. Then, $C(\hat{A}, \hat{A}, |\psi\rangle) = \sum_{i=1}^{r} C(\hat{A}_i, \hat{A}_i, |\psi_i\rangle)$, which means that if $|\psi\rangle$ has $p = 2$, there exists at least one $|\psi_i\rangle$ which has $p = 2$ and $|Z_i| = O(N)$. Without loss of generality, we assume that $|\psi_1\rangle$ has $p = 2$ and $|Z_1| = O(N)$. Let us consider the Schmidt decomposition of $|\psi_1\rangle$ between a single site $l \in Z_1$ and the rest of it $Z_1 - l$:

$$|\psi_1\rangle = \sqrt{\lambda_0}|\xi_0\rangle_l|\eta_0\rangle_{Z_1-l} + \sqrt{\lambda_1}|\xi_1\rangle_l|\eta_1\rangle_{Z_1-l}.$$

Without loss of generality, we assume $\lambda_0 \geq \lambda_1$. Since $|\psi_1\rangle$ is inseparable by assumption, $|\eta_0\rangle_{Z_1-l}$ and $|\eta_1\rangle_{Z_1-l}$ have no common factor. Let us define local operators on site m ($m \in Z_1$) as $\hat{t}_x(m) \equiv |\xi_0\rangle\langle\xi_1| + |\xi_1\rangle\langle\xi_0|$, $\hat{t}_y(m) \equiv -i|\xi_0\rangle\langle\xi_1| + i|\xi_1\rangle\langle\xi_0|$, and $\hat{t}_z(m) \equiv |\xi_0\rangle\langle\xi_0| - |\xi_1\rangle\langle\xi_1|$. Any local operator on site m is written as $\hat{a}(m) = \sum_{\alpha=x,y,z} c_{\alpha,m}\hat{t}_\alpha(m)$.

By some calculation, we can show $C(\hat{t}_\alpha(l), \hat{t}_\beta(l'), |\psi_1\rangle) \leq \sqrt{4\lambda_0\lambda_1}$ for $\alpha, \beta = x, y, z$ and $l' \in Z_1 - l$. Since $E(l) \equiv -\lambda_0 \log_2 \lambda_0 - \lambda_1 \log_2 \lambda_1 \geq 2\min(\lambda_0, \lambda_1) = 2\lambda_1$, we obtain $C(\hat{t}_\alpha(l), \hat{t}_\beta(l'), |\psi_1\rangle) \leq \sqrt{4\lambda_0\lambda_1} \leq \sqrt{4\lambda_1} \leq \sqrt{2E(l)}$. Let us assume that $\sum_{l\in Z_1} \sqrt{2E(l)} = o(N)$. Then,

$$
\begin{aligned}
C(\hat{A}, \hat{A}, |\psi_1\rangle) &= \sum_{l\in Z_1} \sum_{l'\in Z_1 - l} \sum_{\alpha,\beta} c_{\alpha,l} c_{\beta,l'} C(\hat{t}_\alpha(l), \hat{t}_\beta(l'), |\psi_1\rangle) \\
&\quad + \sum_{l\in Z_1} \sum_{\alpha,\beta} c_{\alpha,l} c_{\beta,l} C(\hat{t}_\alpha(l), \hat{t}_\beta(l), |\psi_1\rangle) \\
&\leq \big(|Z_1| - 1\big) \sum_{l\in Z_1} \sqrt{2E(l)} + |Z_1| \\
&= o(N^2),
\end{aligned}
$$

which means that $|\psi_1\rangle$ has $p < 2$. Since it contradicts to the assumption, we obtain $\sum_{l\in Z_1} \sqrt{2E(l)} = O(N)$. Since $0 \leq E(l) \leq 1$, this means that the number of $l \in Z_1$ such that $E(l) = O(1)$ is $O(N)$.

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[1] E. Schrödinger, Naturwissenschaften **23**, 807 (1935).
[2] E. M. Chudnovsky and J. Tejada, *Macroscopic Quantum Tunneling of the Magnetic Moment* (Cambridge University Press, Cambridge, 1998).
[3] D. Leibfried, et. al., Nature **438**, 639 (2005).
[4] A. J. Leggett, Prog. Theor. Phys., Suppl. **69**, 80 (1980).
[5] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[6] W. Dür, et. al., Phys. Rev. Lett **89**, 210402 (2002).
[7] J. I. Korsbakken, et. al., Phys. Rev. A **75**, 042106 (2007).
[8] F. Marquardt, et. al., Phys. Rev. A **78**, 012109 (2008).
[9] A. Shimizu and T. Miyadera, Phys. Rev. Lett. **89**, 270403 (2002).
[10] A. Shimizu and T. Morimae, Phys. Rev. Lett. **95**, 090401 (2005).
[11] R. Horodecki, et. al., Rev. Mod. Phys. **81**, 865 (2009).
[12] L. Amico, et. al., Rev. Mod. Phys. **80**, 517, (2008).
[13] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[14] F. Verstraete, et. al., Phys. Rev. Lett. **92**, 027901 (2004).
[15] V. Vedral, et. al., Phys. Rev. Lett. **78**, 2275 (1997).
[16] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[17] M. Koashi, et. al., Phys. Rev. A **62**, 050302 (2000).
[18] M. Ohya and D. Petz, *Quantum entropy and its use* (Springer-Verlag, Berlin 1993).
[19] Since the logarithmic robustness is larger than the relative entropy entanglement for pure states [20], the logarithmic robustness is also large if $p = 2$.
[20] M. Hayashi, et. al., Phys. Rev. Lett. **96**, 040501 (2006).
[21] D. A. Meyer and N. R. Wallach, J. Math. Phys. **43**, 4273 (2002).
[22] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[23] Otfried Gühne and G. Toth, Phys. Rep. **474**, 1 (2009).
[24] D. T. Pope and G. J. Milburn, Phys. Rev. A **67**, 052107 (2003).
[25] M. Hein, et. al., Phys. Rev. A **69**, 062311 (2004).
[26] H. Barnum and N. Linden, J. Phys. A **34**, 6787 (2001).